Theory of the evolutionary minority game

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We present a theory describing a recently introduced model of an evolving, adaptive system in which agents compete to be in the minority. The agents themselves are able to evolve their strategies over time in an attempt to improve their performance. The theory explicitly demonstrates the self-interaction, or *market impact*, that agents in such systems experience.

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Agent-based models of complex adaptive systems [1] have attracted much attention recently since they provide invaluable insight into the nontrivial global behavior of a population of competing agents and have potential applications in the ''physics of finance'' [2–4]. Challet and Zhang [5] (also see Ref. [6]) proposed the minority game (MG), in which agents successively compete to be in the minority by making decisions based on global information created by the agents themselves. Nontrivial fluctuations arising in the game [5,6] can be understood in terms of the dynamical formation of crowds and anticrowds comprising of agents using anticorrelated pairs of strategies [7]. The game was also related formally to spin glasses [8]. However, the MG does not incorporate evolution. Agents are stuck with their initial strategies. In the real world, agents tend to evolve by learning from past experience, and will also stop playing disastrous strategies. Recently, we proposed the evolutionary minority game (EMG) which allows for an *evolving* population [9–11]. Here we provide a theory for the EMG, yielding good agreement with numerical data and correctly including the self-interaction of the agents.

The EMG [9] consists of an odd number N of agents repeatedly choosing to be in room 0 (e.g., sell) or room 1 (e.g., buy). The winners are those in the minority room. A single binary digit denotes the winning minority room. The agents have a common ''memory'' look-up table containing the outcomes from the most recent occurrences of all 2^m bit strings of length m. Faced with a given bit string of recent occurrences, each agent chooses the outcome in the memory with probability p, which we refer to as the agent's ''gene'' value, and chooses the opposite action with probability $1-p$. To incorporate evolution, we assign $+1(-1)$ point to every agent in the minority (majority) room at each time step. If an agent's score falls below a value d $(d<0)$, a new p value is chosen randomly within a range R centered on the old p. We impose reflective boundary conditions to ensure that $0 \le p \le 1$, although our conclusions do not depend on this particular choice of boundary conditions.

We focus on two quantities, $P(p)$ and $L(p)$, in the long time limit. Here $P(p)$ is the frequency distribution of gene values, typically taken in the long time limit over a time window and normalized to unity; $L(p)$ is the lifespan defined as the average length of time a gene value p survives between modifications [9]. Figure 1 shows $L(p)$ and $P(p)$ (inset) for a range of m values. Both $P(p)$ and $L(p)$ are symmetric about $p=1/2$, with peaks around $p=0$ and 1. The

results are insensitive to the initial distribution of p values. Surprisingly, agents who either always follow or never follow what happened last time, generally perform better than cautious agents using an intermediate value of p. There is no explicit dependence on m for $P(p)$ and $L(p)$ [9,11,12]. D'Hulst and Rodgers [13] proposed an analytic theory giving a $P(p)$ somewhat similar to Fig. 1. However, the theory was developed based on a model in which agents use different memory look-up tables: the corresponding $P(p)$ is then m dependent [11], in contrast to the EMG results shown in Fig. 1.

The basic idea of the present formulation is to consider the interaction between a particular agent and the rest of the population. Consider a certain moment of the game in the steady-state regime. Let the predictor in the memory look-up table be 1; i.e., go to room 1. As long as the winning room is defined as the minority room, i.e. with a cutoff at $(N-1)/2$, the following arguments also hold if the predictor says 0. Let $F_N(n)$ be the probability of the attendance being n in the predicted room. From the central limit theorem, $F_N(n)$ is approximately a Gaussian distribution with mean $N\bar{p}$ and variance $N\int_0^1 P(p)p(1-p)dp$. Here $\bar{p}=\int_0^1 pP(p)dp$, which is known if the distribution $P(p)$ is known. However, $P(p)$
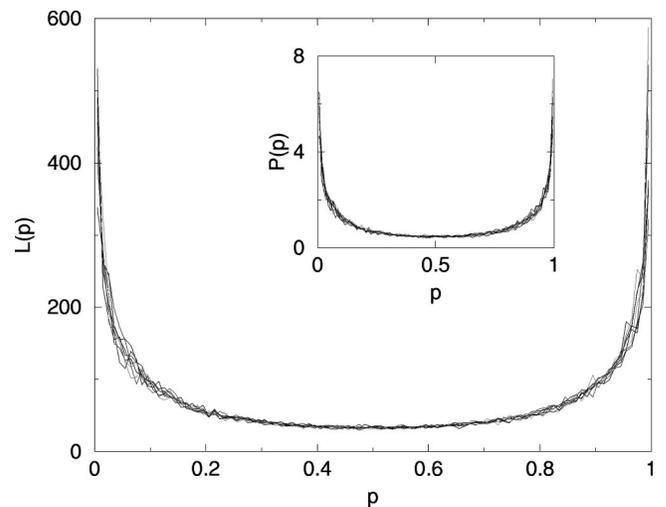

FIG. 1. The life span $L(p)$ as a function of gene value p for $m=1,2,\dots,8$. The inset shows the distribution of gene values $P(p)$ for different m values. Both $L(p)$ and $P(p)$ are insensitive to m. Parameters are $N=101$, $d=-4$, and $R=0.2$. The quantities shown in Figs. 1–4 are dimensionless.

is the unknown that we are going to solve for. In the steady state, $F_N(n)$ becomes identical to the probability of the attendance in any one of the two rooms since the two outcomes are equally likely. Consider the action of a particular agent, say the kth player, in the background of the $N-1$ other agents. Let $G_{N-1}^k(n)$ be the probability of the attendance being n in the predicted room, given that there are only $(N-1)$ agents participating in the game (i.e. excluding the kth agent). Hence

$$F_N(n) = p_k G_{N-1}^k(n-1) + (1-p_k) G_{N-1}^k(n), \qquad (1)$$

where $n \neq 0, N$. Here p_k is the p-value of the kth agent at that moment. The physical meaning of Eq. (1) is transparent. An attendance of n in room 1 is achieved if the attendance by the $(N-1)$ agent background is $n-1$ and the kth agent decides to go to room 1 (first term), or the attendance by the $(N-1)$ agent background is n and the kth agent decides not to go to room 1 (second term).

Let $\tau(p_k)$ be the winning probability of the kth agent. Hence

$$\tau(p_k) = p_k \sum_{n=0}^{(N-3)/2} G_{N-1}^k(n) + (1-p_k) \sum_{n=(N+1)/2}^{N-1} G_{N-1}^k(n), \qquad (2)$$

since the kth agent wins if (i) the attendance is below $(N-3)/2$ in room 1 before he makes his move and he decides to go to room 1; or (ii) the attendance is above $(N+1)/2$ in room 1 before he makes his move and he decides not to go to room 1. The kth agent is only characterized by p_k; hence $\tau(p_k)$ can also be interpreted as the success rate of an agent using gene value p_k. Following Eq. (1),

$$\sum_{n=1}^{(N-3)/2} F_N(n) = \sum_{n=1}^{(N-3)/2} G_{N-1}^k(n) + p_k G_{N-1}^k(0)$$
$$- p_k G_{N-1}^k\left(\frac{N-3}{2}\right).$$

Since $F_N(0) = (1-p_k) G_{N-1}^k(0)$, which follows from the consideration that room 1 is empty only if the other $N-1$ agents do not go to room 1 and the kth agent does not go to room 1, we have

$$\sum_{n=0}^{(N-3)/2} G_{N-1}^k(n) = \sum_{n=0}^{(N-3)/2} F_N(n) + p_k G_{N-1}^k\left(\frac{N-3}{2}\right). \qquad (3)$$

Similarly,

$$\sum_{n=(N+1)/2}^{N-1} F_N(n) = \sum_{n=(N+1)/2}^{N-1} G_{N-1}^k(n) + p_k G_{N-1}^k\left(\frac{N-1}{2}\right)$$
$$- p_k G_{N-1}^k(N-1),$$

and $F_N(N) = p_k G_{N-1}^k(N-1)$; therefore,

$$\sum_{n=(N+1)/2}^{N-1} G_{N-1}^k(n) = \sum_{n=(N+1)/2}^{N} F_N(n) - p_k G_{N-1}^k\left(\frac{N-1}{2}\right). \qquad (4)$$
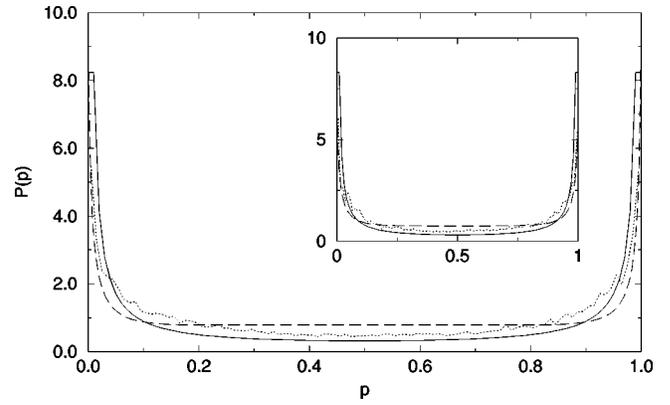

FIG. 2. The frequency distribution $P(p)$ for $N=101$ and $N=51$ (inset). $d=-4$ and $R=0.2$. Dotted lines are numerical simulation data. Solid lines are results of the present theory. Dashed lines are the theory of Ref. [13].

Substituting Eqs. (3) and (4) into Eq. (2), and using Eq. (1) to express $G_{N-1}^k[(N-3)/2]$ in terms of $G_{N-1}^k[(N-1)/2]$ and $F_N[(N-1)/2]$, we obtain

$$\tau(p_k) = p_k \sum_{n=0}^{(N-1)/2} F_N(n) + (1-p_k) \sum_{n=(N+1)/2}^{N} F_N(n)$$
$$- 2p_k(1-p_k) G_{N-1}^k\left(\frac{N-1}{2}\right). \qquad (5)$$

Equation (5) separates $\tau(p_k)$ into three terms, each of which has a physically transparent interpretation. Consider an ''outsider,'' i.e., someone whose action does not affect the outcome but instead is only betting on which side is the winning room using the probability p_k. His winning probability is given by the first two terms in Eq. (5). The third term gives the difference in the winning probability between an outsider from the game and an agent who actually participates in the game. This term is negative, since an agent has a smaller probability of winning when participating in the game. Consider the case in which the background population is split evenly between room 0 and room 1: the kth agent loses no matter what action he takes. Thus the third term represents this self-interaction term, or so-called *market impact* in financial market terminology. The $p_k(1-p_k)$ factor implies that the winning probability increases as p_k deviates further from the value 1/2, and it produces the symmetry about $p=1/2$ in $L(p)$ and $P(p)$ (see Fig. 1). Equation (5) also applies when the predictor says 0: it is independent of the dynamics of the predictor which is determined by the time evolution of outcomes. This further implies that the resulting $P(p)$ and $L(p)$ do not depend on the value of m. For the present EMG, there is no *a priori* perferred room: hence 0 and 1 occur similar numbers of times on average. In this case, the summations in the first and second terms of Eq. (5) in the steady state yield 1/2, and hence $\tau(p_k) = 1/2 - 2p_k(1-p_k) G_{N-1}^k[(N-1)/2]$.

The right hand side of Eq. (5) can be expressed entirely in terms of the function F. From Eq. (1),

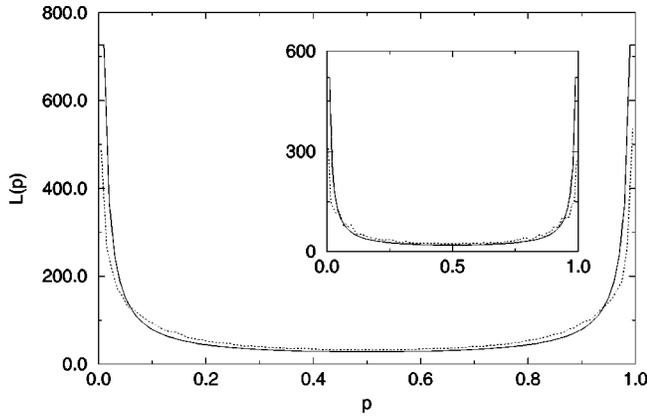$$p_k G_{N-1}^k(n-2) + (1-p_k) G_{N-1}^k(n-1) = F_N(n-1). \qquad (6)$$

FIG. 3. The lifespan $L(p)$ for $N=101$ and 51 (inset). Dotted lines are numerical simulation data. Solid lines are the present theory. Other parameters are the same as Fig. 2.

Multiplying Eqs. (1) and (6) by $(1-p_k)$ and p_k, respectively, and subtracting the resulting equations, yields

$$(1-p_k)F_N(n)-p_kF_N(n-1)=(1-p_k)^2G_{N-1}^k(n)$$
$$-p_k^2G_{N-1}^k(n-2).$$

Repeatedly applying Eq. (1), we can eliminate $G_{N-1}^k(n-2), G_{N-1}^k(n-3), \ldots$, to obtain

$$\sum_{j=0}^{n}(-1)^{n-j}F_N(j)\left(\frac{p_k}{1-p_k}\right)^{n-j}=(1-p_k)G_{N-1}^k(n). \quad (7)$$

Similarly, we can obtain

$$\sum_{j=n+1}^{N}(-1)^{j-n-1}F_N(j)\left(\frac{1-p_k}{p_k}\right)^{j-n-1}=p_kG_{N-1}^k(n). \quad (8)$$

The results are exact; however, it makes sense to use Eq. (7) for small p_k and Eq. (8) for $p_k\sim1$. Using Eqs. (7) or (8) for $n=(N-1)/2$, and substituting into Eq. (5), we obtain $\tau(p_k)$ entirely in terms of $F_N(n)$, and the label k becomes irrelevant. Since $\tau(p_k)$ can be regarded as the winning probability of an agent who is using a gene value p, we henceforth denote it by $\tau(p)$. Reference [13] notes that the stationary distributions $P(p)$ and $L(p)$ are proportional to each other, $P(p)\propto L(p)$, with the proportionality constant being independent of p. This relation follows from the balance between the fluxes of agents into and out of a region in p space in the steady state. Since an agent using p loses $(1-2\tau(p))$ points each turn on average [13], the lifespan $L(p)$ is given by $\tau(p)=|d|/(1-2\tau(p))$. Hence

$$P(p)\propto\frac{1}{1/2-\tau(p)}, \quad (9)$$

with the proportionality constant determined by the normalization $\int_0^1 P(p)dp=1$.

It is straightforward to construct an iterative calculation scheme for $P(p)$ as follows: (a) assume a form for $P(p)$; (b) obtain $F_N(n)$ by evaluating $\bar{p}$ and the standard deviation from $P(p)$; (c) use Eq. (5) together with Eqs. (7) and (8) to
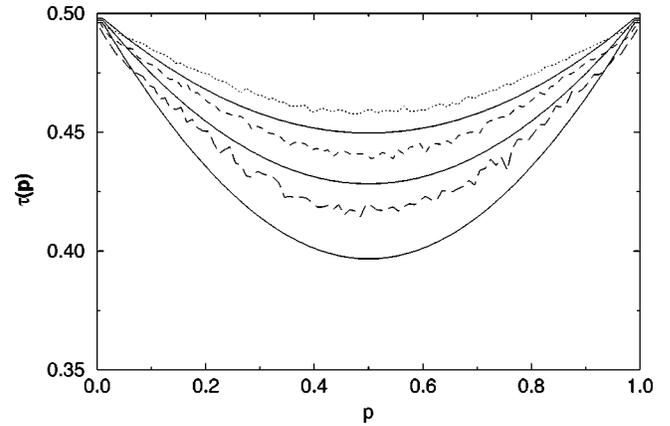

FIG. 4. The winning probability $\tau(p)$ for different values of N. Solid lines are the present theory, while dotted and dashed lines are from numerical simulations. The three sets of lines from top to bottom at $p=1/2$ correspond to $N=201$, 101, and 51, respectively. Other parameters are the same as in Fig. 2.

obtain $\tau(p)$; (d) calculate $P(p)$ from $\tau(p)$ using Eq. (9) and the normalization condition; and (e) check for convergence of $P(p)$ and, if necessary, repeat the steps until convergence. Note that Eq. (5) is employed since it is valid for all forms of initial guess for $P(p)$, including those which are nonsymmetrical about $p=1/2$.

Results for $P(p)$ and $L(p)$ obtained by carrying out the calculation scheme are shown in Figs. 2 and 3, together with results of numerical simulation for $N=51$ and 101. When properly normalized, $P(p)$ is not sensitive to N, while $L(p)$ depends on N. Results from our theory are in good agreement with numerical data. The results for $P(p)$ as obtained in Ref. [13] are also shown in Fig. 2 for comparison: note that the results of Ref. [13] show a plateau over a significant range of p in contrast to the present theory and the numerical simulations. The results from the present theory are hence in better agreement with the numerical results. As further evidence for the validity of our theory, results for $\tau(p)$ as a function of p for $N=51$, 101, and 201 are shown in Fig. 4. The numerical data are found by counting the number of times an agent with p wins. Note that $\tau(p)$ provides a better test than $P(p)$ for the validity of any theory, since many forms of $\tau(p)$ can give rise to similar forms for $P(p)$. In contrast to the present theory and numerical results, the expression for $\tau(p)$ in Ref. [13] gives a very small $\tau(p)$ for a significant range of p around $p=1/2$ corresponding to the plateau in $P(p)$. Figure 4 shows that the correct $\tau(p)$ in the steady state, which follows from Eq. (5), is indeed consistent with the form $\tau(p)\sim1/2-\mathcal{A}(N)p(1-p)$ where $\mathcal{A}(N)$ is an N-dependent constant which decreases with N as $1/\sqrt{N}$. Such a scaling with N also makes sense from random walk arguments.

In summary, we have presented a mean-field-like theory of the EMG based on the consideration of a particular agent in the environment formed by the rest of the population. The results are in good agreement with numerical data. The present theory can be readily generalized to different variations [14,15] of the EMG, e.g., when the winning decision is assigned according to whether the attendance is lower than a certain cutoff [14].

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